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1985 J. Phys. A: Math. Gen. 18 L235

(http://iopscience.iop.org/0305-4470/18/5/002)

J. Phys. A: Math. Gen. 18 (1985) L235-L239. Printed in Great Britain

LETTER TO THE EDITOR

Path integral on S^2: The Rosen-Morse oscillator

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Received 23 January 1985

Abstract. Guided by the group chain $SO(3) \supset O(2)$, we construct an angular path integral for the symmetric Rosen-Morse oscillator on S^2. By explicit path integration, we obtain the normalised energy eigenfunction as well as the exact energy spectrum.

In recent years, some useful techniques have been developed in path integral calculation. For example, the change of variables in a path integral is not all practical, but the time rescaling trick has made coordinate transformations more widely applicable. With the aid of such new techniques, the listing of exactly path-integrable examples has been increased, which includes the Aharonov-Bohm effect (Inomata and Singh 1978, Bernido and Inomata 1981), the hydrogen atom (Duru and Kleinert 1979, Ho and Inomata 1982, Inomata 1984), the entanglement probability of macromolecules (Tanikella and Inomata 1982), the Morse oscillator (Cai *et al* 1983, Duru 1983), the Dirac-Coulomb problem (Kayed and Inomata 1984) and the charge-monopole system (Dürr *et al* 1984). Now we are generally able to evaluate a path integral if it is intrinsically reducible in the short time limit to a confluent hypergeometric equation. In this connection, it is interesting to point out that most of the examples so far known as path-integrable are of $SO(n) \times SO(2, 1)$ symmetry ($n \leqslant 3$).

Recently, from various aspects (Brezin *et al* 1977, Yoon and Negele 1977, Nieto 1978, Alhassid *et al* 1983, 1984, Frank and Wolf 1984), there has been renewed interest in the Rosen-Morse potential (Rosen and Morse 1932) of the symmetric form,

$$V(x) = -B \operatorname{sech}^2 ax \tag{1}$$

where a and B are positive constants. Apparently, the one-dimensional cartesian path integral for this potential is not Gaussian. The Schrödinger equation with (1) is not reducible to a confluent hypergeometric equation. Certainly this does not belong to the current list of path-integrable examples. However, the group theoretical analysis (Alhassid *et al* 1983, 1984, Frank and Wolf 1984) indicates that a group chain relevant to this system is $SO(3) \supset O(2)$. In the present paper, guided by the group chain, we propose a new method of evaluating Feynman's path integral for the symmetric Rosen-Morse oscillator—the bound states in (1). First, we construct a path integral on $S^2 = SO(3)/O(2)$ for the Green function of the oscillator with the Lagrangian

$$L = \tfrac{1}{2} m\dot{x}^2 + B \operatorname{sech}^2 ax \tag{2}$$

where we set $B = \lambda(\lambda - 1)(\hbar^2 a^2/2m)$ with $\lambda > 1$. Then we calculate the path integral explicitly to find the energy spectrum and the normalised energy eigenfunctions consistent with those derived from the Schrödinger equation (Nieto 1978). The con-

0305-4470/85/050235+05\$02.25

structed path integral on S^2 is a special case of the polar coordinate path integral considered earlier (Edwards and Gulyaev 1964, Peak and Inomata 1969).

Let us start by writing Feynman's propagator in the form,

$$K(x'', x'; t'', t') = (2\pi\hbar)^{-1} \iint P(x'', x'; \tau) \exp[-\mathrm{i}E(t''-t')/\hbar]\, \mathrm{d}\tau\, \mathrm{d}E, \tag{3}$$

where

$$P(x'', x'; \tau) = \int \exp\left((\mathrm{i}/\hbar) \int^{\tau} (L+E)\, \mathrm{d}t\right) \mathrm{d}x. \tag{4}$$

The corresponding Green function in the energy representation is

$$G(x'', x' : E) = (\mathrm{i}\hbar)^{-1} \int P(x'', x'; \tau)\, \mathrm{d}\tau. \tag{5}$$

The path integral (4) for the Lagrangian (2) may be written on the sliced time basis as

$$P(x'', x'; \tau) = \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\left(\frac{\mathrm{i}}{\hbar} W_j\right) \prod_{j=1}^{N} \left(\frac{m}{2\pi\mathrm{i}\hbar\tau_j}\right)^{1/2} \prod_{j=1}^{N-1} \mathrm{d}x_j \tag{6}$$

with a modified short time action,

$$W_j = (m/2\tau_j)(\Delta x_j)^2 + B\tau_j \operatorname{sech} ax_j \operatorname{sech} ax_{j-1} + E\tau_j, \tag{7}$$

where $x_j = x(t_j)$, $\Delta x_j = x_j - x_{j-1}$, $\tau_j = t_j - t_{j-1}$ and $\tau = \Sigma^N \tau_j$. Note that in (7) the terms of $\mathrm{O}(\tau_j^2)$ have been ignored as usual.

Evidently, (6) is not integrable by the x-variable. In an effort to reduce (6) into an integrable form, we transform $x_j \in (-\infty, \infty)$ into an angular variable $\theta_j \in [0, \pi)$ and the local time interval σ_j by

$$\operatorname{sech} ax_j = \sin\theta_j, \qquad \tau_j = \sigma_j / \sin^2 \hat{\theta}_j \tag{8}$$

where $\sin^2\hat{\theta}_j = \sin\theta_j \sin\theta_{j-1}$. In the new variables, (6) becomes

$$P(x'', x'; \tau) = a(\sin\theta' \sin\theta'')^{1/2} \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\left(\frac{\mathrm{i}}{\hbar} W_j\right) \prod_{j=1}^{N} \left(\frac{m}{2\pi\mathrm{i}\hbar a^2 \sigma_j}\right)^{1/2} \prod_{j=1}^{N-1} \mathrm{d}\theta_j \tag{9}$$

with

$$W_j = \frac{m(\Delta\theta)^2}{2a^2\sigma_j} - \frac{m(\Delta\theta)^4}{24a^2\sigma_j} - \frac{m(\Delta\theta)^4}{24a^2\sigma_j \sin^2\hat{\theta}} + \frac{E\sigma_j}{\sin^2\hat{\theta}} + B\sigma_j. \tag{10}$$

In the above, we have suppressed the subscript j of θ, and hereafter whenever appropriate we shall do the same for θ and others.

The transformed action (10) is by no means simpler. The first step of simplifying (10) is to replace the third term by an equivalent one, $\hbar^2 a^2 \sigma_j/(8m \sin^2\hat{\theta})$. This can be justified by the relation valid for large $\alpha > 0$,

$$\int_0^c y^{2n} \exp[-\alpha y^2 + \beta y^4 + \beta' y^4 + \mathrm{O}(y^6)]\, \mathrm{d}y$$

$$= \int_0^c y^{2n} \exp[-\alpha y^2 + \beta y^4 + \tfrac{3}{4}\beta' \alpha^{-2} + \mathrm{O}(\alpha^{-3})]\, \mathrm{d}y, \tag{11}$$

where c is a constant and n a positive integer. For $c \to \infty$, (11) is quite obvious (Cai *et al* 1983). Noticing that

$$\int_0^c \exp(\mathrm{i}\alpha y^2) f(y)\,\mathrm{d}y = \alpha^{-1/2} \int_0^{\alpha^{-1/2}c} \exp(\mathrm{i}\alpha y^2) f(\alpha^{-1/2} y)\,\mathrm{d}y,$$

we can assure that (11) is valid for any c insofar as α is large. We also remind ourselves that in a path integral $(\Delta t)^{-1}\cos(\Delta\theta) = [1 - \frac{1}{2}(\Delta\theta)^2 + (\Delta\theta)^4/24]/(\Delta t)$ is a valid approximation for an angular variable (Edwards and Gulyaev 1964). Thus we can write the action (10) in a simpler form,

$$W_j = (m/a^2\sigma_j)[1 - \cos(\Delta\theta)] + (\sigma_j/\sin^2\hat{\theta})[E + (\hbar^2 a^2/8m)] + B\sigma_j. \tag{12}$$

The path integral (9), even having (12), is not yet ready for integration. We have to go one more step further by introducing another angular variable $\phi \in [0, 2\pi]$. Namely, we put the second term of (12), multiplied by $(\mathrm{i}/\hbar)$ and exponentiated, into the form,

$$\begin{aligned} &\exp\{(\mathrm{i}\sigma_j/\hbar\sin^2\hat{\theta})[E + (\hbar^2 a^2/8m)]\} \\ &\quad = (m\sin^2\hat{\theta}/8\pi\mathrm{i}\hbar a^2\sigma_j)^{1/2} \int_{-2\pi}^{2\pi} \exp[(\mathrm{i}m\sin^2\hat{\theta}/\hbar a^2\sigma_j) \\ &\quad\quad \times (1 - \cos\Delta\phi) + \mathrm{i}k\Delta\phi]\,\mathrm{d}(\Delta\phi) \end{aligned} \tag{13}$$

where $k = (-2mE/\hbar^2 a^2)^{1/2}$. For this, we have used the approximation formula for large z and an integer k,

$$\int_{-2\pi}^{2\pi} \exp[\mathrm{i}k\varphi - z(1 - \cos\varphi)]\,\mathrm{d}\varphi = (8\pi/z)^{1/2} \exp[-(k^2 - \tfrac{1}{4})/2z]. \tag{14}$$

With the help of an integral representation and the asymptotic formula of the modified Bessel function (Langguth and Inomata 1979),

$$I_k(z) = (2\pi z)^{-1/2} \exp[z - (k^2 - \tfrac{1}{4})/2z], \tag{15}$$

we can easily derive (14). Moreover, noticing that for $f(\Delta\phi + 2\pi) = f(\Delta\phi)$

$$\int_{-2\pi}^{2\pi} f(\Delta\phi)\,\mathrm{d}(\Delta\phi) = 2\int_0^{2\pi} f(\Delta\phi)\,\mathrm{d}\phi, \tag{16}$$

we combine (12) and (13) together to obtain

$$\begin{aligned} \exp(\mathrm{i}W_j/\hbar) &= (m\sin^2\hat{\theta}/2\pi\mathrm{i}\hbar a^2\sigma_j)^{1/2} \exp(\mathrm{i}B\sigma_j/\hbar) \\ &\quad \times \int_0^{2\pi} \exp(\mathrm{i}k\,\Delta\phi_j) \exp[\mathrm{i}m(1 - \cos\Theta)/\hbar a^2\sigma_j]\,\mathrm{d}\phi_j \end{aligned} \tag{17}$$

where $\cos\Theta = \cos\theta_j \cos\theta_{j-1} + \sin\theta_j \sin\theta_{j-1}\cos(\Delta\phi_j)$. Substitution of (17) into (9) yields

$$P(x'', x'; \tau) = a\sin\theta' \sin\theta'' \exp(\mathrm{i}B\sigma/\hbar) \int_0^{2\pi} Q(\theta'', \theta'; \phi''; \sigma)\,\mathrm{d}\phi'' \tag{18}$$

where $\sin\theta' = \operatorname{sech} ax''$, $\sin\theta'' = \operatorname{sech} ax''$, $\phi' = 0$ and $\sigma = \tau \operatorname{sech} ax' \operatorname{sech} ax''$. The integrand of (18) is indeed a path integral on S^2,

$$Q(\theta'', \theta'; \phi''; \sigma) = \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp(\mathrm{i}\tilde{W}_j/\hbar) \prod_{j=1}^{N} (m/2\pi\mathrm{i}\hbar a^2\sigma_j) \prod_{j=1}^{N-1} \sin\theta_j\,\mathrm{d}\theta_j\,\mathrm{d}\phi_j \tag{19}$$

with

$$\tilde{W}_j = (m/a^2\sigma_j)(1-\cos\Theta_j) + \hbar k\,\Delta\phi_j, \tag{20}$$

which is now integrable (Edwards and Gulyaev 1964, Peak and Inomata 1969). The Green function (5) and the propagator (3) are therefore evaluated, respectively, by

$$G(x'', x'; E) = (a/\mathrm{i}\hbar)\int_{-\infty}^{\infty}\int_0^{2\pi} Q(\theta'', \theta'; \phi''; \sigma)\exp(\mathrm{i}B\sigma/\hbar)\,\mathrm{d}\phi''\,\mathrm{d}\sigma \tag{21}$$

and

$$K(x'', x'; t'', t') = (\mathrm{i}/2\pi)\int_{-\infty}^{\infty} G(x'', x'; E)\exp[-\mathrm{i}E(t''-t')/\hbar]\,\mathrm{d}E. \tag{22}$$

The path integration of (19) on S^2 is rather straightforward. Employing the standard expansion formula,

$$\exp(u\cos\Theta_j) = (\pi/2u)^{1/2}\sum_{l=0}^{\infty}\sum_{\mu=-l}^{l}(2l+1)[\Gamma(l-\mu+1)/\Gamma(l+\mu+1)]$$
$$\times I_{l+1/2}(u)\exp(\mathrm{i}\mu\,\Delta\phi_j)P_l^{\mu}(\cos\theta_j)P_l^{\mu}(\cos\theta_{j-1}), \tag{23}$$

and the asymptotic relation (15), we get for (20)

$$\exp(\mathrm{i}\tilde{W}_j/\hbar) = (\mathrm{i}\hbar a^2\sigma_j/m)\sum_{l=0}^{\infty}\sum_{\mu=-l}^{l}[(l+\tfrac{1}{2})\Gamma(l-\mu+1)/\Gamma(l+\mu+1)]$$
$$\times\exp[-\mathrm{i}l(l+1)\hbar a^2\sigma_j/2m]\exp[\mathrm{i}(\mu+k)\Delta\phi_j]P_l^{\mu}(\cos\theta_j)P_l^{\mu}(\cos\theta_{j-1}). \tag{24}$$

Here also the j subscripts of l and μ are suppressed. With (24), we can readily carry out the angular integrations in (19) by using the orthogonality relations of the exponential functions and the associated Legendre functions. As a result of the angular integrations, we find

$$Q(\theta'', \theta'; \phi''; \sigma) = \sum_{l=0}^{\infty}\sum_{\mu=-l}^{l}[(l+\tfrac{1}{2})\Gamma(l-\mu+1)/2\pi\Gamma(l+\mu+1)]$$
$$\times\exp[-\mathrm{i}l(l+1)\hbar a^2\sigma/2m]\exp[\mathrm{i}(\mu+k)\phi'']P_l^{\mu}(\cos\theta')P_l^{\mu}(\cos\theta'') \tag{25}$$

where $l = l''$ and $\mu = \mu''$. Now, inserting this into (21), completing the remaining integrations and summing over μ, we arrive at

$$G(x'', x'; E) = (a/\mathrm{i}\hbar)[2\pi m/\hbar a^2(\lambda-\tfrac{1}{2})]\sum_{l=0}^{\infty}[(l+\tfrac{1}{2})\Gamma(l+k+1)/\Gamma(l-k+1)]$$
$$\times\delta(\lambda-l-1)P_l^{-k}(\cos\theta')P_l^{-k}(\cos\theta'') \tag{26}$$

where we have set $B = \lambda(\lambda-1)\hbar^2a^2/2m$. As the terms for $l<k$ vanish due to the Γ function in the denominator, we set $n = l-k$ to rewrite (26) in the form,

$$G(x'', x'; E) = (2\pi m/\mathrm{i}\hbar^2 a)\sum_{n=0}^{\infty}[\Gamma(2\lambda-n-1)/\Gamma(n+1)]$$
$$\times\delta(\lambda-n-k-1)P_{\lambda-1}^{n-\lambda-1}(\tanh ax')P_{\lambda-1}^{n-\lambda-1}(\tanh ax''). \tag{27}$$

Since $k = (-2mE/\hbar^2a^2)^{1/2}$ and hence $\delta(\lambda-n-k-1) = (\hbar^2a^2/m)(\lambda-n-1)\times$

$\delta[E+(\lambda-n-1)^2\hbar^2a^2/2m]$, the E integration of (22) yields

$$K(x'', x'; t'', t') = \sum_{n=0}^{\infty} \psi_n^*(x')\psi_n(x'') \exp[-iE_n(t''-t')/\hbar] \qquad (28)$$

where

$$E_n = -(\hbar^2a^2/2m)(\lambda-n-1)^2 \qquad n=0,1,2,\ldots \leqslant \lambda-1 \qquad (29)$$

$$\psi_n = [a(\lambda-n-1)\Gamma(2\lambda-n-1)/\Gamma(n+1)]^{1/2} P_{\lambda-1}^{n-\lambda+1}(\tanh ax) \qquad (30)$$

which are the exact energy spectrum of the symmetric Rosen–Morse oscillator and the corresponding normalised wavefunction consistent with those obtained from the Schrödinger equation (Nieto 1978). Although our calculation has been made for λ = integer>1, it is easy to continue (28) analytically for λ = non-integer>1. One of the advantages of the path integral calculation is that the resultant propagator, satisfying the limiting condition $K(x'', x'; t''\to t') = \delta(x''-x')$, leads naturally to the wavefunction with a correct normalisation factor. The present angular path integration method can be applied to the symmetric Pöschl–Teller oscillator as well.

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